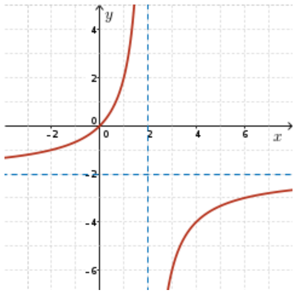
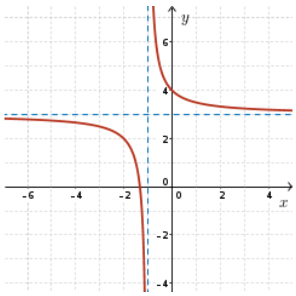
Rational Functions of the Form $y = \frac{ax+b}{cx+d}$

Exercises

1. For each rational function of the form $y = \dfrac{ax+b}{cx+d}$, determine the equation of any horizontal and vertical asymptotes. State the coordinates of any point of discontinuity (hole), if one exists.
   a. $y = \dfrac{3}{x+1}$
   b. $y = \dfrac{x}{2x-3}$
   c. $y = \dfrac{-2x+2}{x-1}$
   d. $y = \dfrac{-2x+3}{x-4}$
2. For each function, identify the domain, intercepts, asymptotes and any points of discontinuity. Investigate the behaviour of the graph of the function near its asymptotes. Graph the function. Identify the intervals where the function is positive or negative, and where the function is increasing or decreasing.
   a. $f(x) = \dfrac{5}{3x-9}$
   b. $g(x) = \dfrac{-2x}{4x+8}$
   c. $h(x) = \dfrac{2x-2}{x+2}$
   d. $f(x) = \dfrac{2x-3}{5-x}$
3. For each of the following functions, convert the equation from the form $y = \dfrac{ax+b}{cx+d}$ to the form $y = \dfrac{a}{b(x-h)} + k$ or vice versa.
   a. $y = \dfrac{2x}{x-1}$
   b. $y = \dfrac{4x+3}{x+2}$
   c. $y = \dfrac{-1}{x+3} + 2$
   d. $y = \dfrac{-3x+5}{2x-4}$
   e. $y = \dfrac{4}{2x+3} - 1$
4. For each of the functions $f(x) = \dfrac{2x+7}{x+4}$ and $g(x) = \dfrac{x}{x+2}$,
   a. Identify the domain, intercepts and asymptotes of the function and sketch its graph.
   b. Express the equation of the function in the form $y = \dfrac{a}{b(x-h)} + k$ and verify the graph obtained in part a) by applying transformations to the graph of $y = \dfrac{1}{x}$.
5. Determine a possible equation of a rational function of the form $f(x) = \dfrac{ax+b}{cx+d}$ that satisfies the given set of conditions.
   a. Vertical asymptote at $x = 2$, horizontal asymptote at $y = 0$, no x-intercept, y-intercept of 3
   b. Vertical asymptote at $x = \frac{2}{3}$, horizontal asymptote at $y = -1$, passing through the origin
   c. A hole at $x = 4$, y-intercept of 5
   d. Vertical asymptote at $x = -\frac{1}{3}$, horizontal asymptote at $y = \frac{2}{3}$, y-intercept of 4, decreasing for $\{x \in \mathbb{R} \mid x \neq -\frac{1}{3}\}$
   e. Vertical asymptote at $x = -\frac{1}{3}$, horizontal asymptote at $y = \frac{2}{3}$, increasing for $\{x \in \mathbb{R} \mid x \neq -\frac{1}{3}\}$
6. Determine an equation for the rational function, of the form $f(x) = \dfrac{ax+b}{cx+d}$, given the graph shown.
   a.



b.



7. Analyze and sketch the graphs of each of the functions $f(x) = \frac{2x+1}{x-3}$, $g(x) = \frac{2x^2-5x-3}{x-3}$ and $h(x) = \frac{2x^2+9x+4}{x^2+x-12}$.

8. A train leaves Waterloo and travels westbound. Due to track maintenance, it takes the train 2 hours to travel the first 120 km. Once it leaves the construction zone, the train travels at 140 km/h for the rest of the trip. Let s represent the average speed, in km/h, over the entire trip and t represent the time, in hours, spent travelling after leaving the construction zone.

   a. Write an equation for s as a function of t. Identify the domain.

   b. Graph the function.

   c. State the equation of any asymptote and describe its meaning in this situation.

   d. If the average speed of the entire trip is 100 km/h, what is the total distance travelled by the train?

9. The model for the percentage, $P(x)$, of a drug in the bloodstream, x hours after it is taken orally, is $P(x) = \frac{7x}{x^2+2}$.

   a. What is an appropriate domain for this function?

   b. What percentage of the drug, to the nearest tenth, is in the person's bloodstream after:

      i. 1 hour?

      ii. 2 hours?

      iii. 4 hours?

      iv. 12 hours?

      v. 24 hours?

   c. Identify any intercepts and asymptotes of the function and predict the shape of its graph. Use graphing technology to verify your prediction.

   d. Describe what happens to the concentration of the drug, in the bloodstream, over 24 consecutive hours. Does this model seem reasonable?

10. a. Sketch a graph of the function $y = \frac{1}{x^2}$.

   b. Use your sketch in part a) and your knowledge of transformations to sketch $y = \frac{1}{(x+2)^2} + 3$.

   c. Use your sketch in part a) and your knowledge of transformations to sketch $y = \frac{-3}{x^2-6x+9} - 2$.

   d. Sketch a graph of the function $y = \frac{-x^2+2x+2}{x^2-2x+1}$.

Rational Functions of the Form $y = \frac{ax+b}{cx+d}$

Partial Solutions

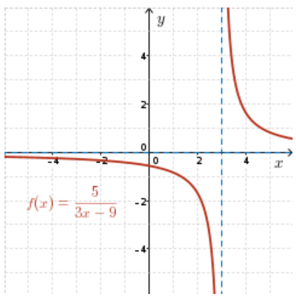
1. There is no solution provided for this question.

2. a. The domain of the function $f(x) = \dfrac{5}{3x-9}$ is $\{x \in \mathbb{R} \mid x \neq 3\}$ since $3x - 9 \neq 0$.

$f(3) = \frac{5}{0}$, an undefined value, so the function has a vertical asymptote of $x = 3$. $f(2.999) \approx -1667$ so $\lim\limits_{x \to 3^-} \dfrac{5}{3x-9} = -\infty$ and $f(3.001) \approx 1667$ so $\lim\limits_{x \to 3^+} \dfrac{5}{3x-9} = \infty$. As $x \to \pm\infty$, $f(x) \to 0$ $\left(\lim\limits_{x \to \pm\infty} \dfrac{5}{3x-9} = 0\right)$ so the horizontal asymptote is $y = 0$.

$f(-1000) \approx -0.0017 < 0$ so the graph approaches $y = 0$ from below as $x \to -\infty$ and $f(1000) \approx 0.0017 > 0$ so the graph approaches $y = 0$ from above as $x \to \infty$.

$f(x)$ does not have an x-intercept because there are no x values that satisfy $0 = \dfrac{5}{3x-9}$. The y-intercept occurs when $x = 0$ so the function crosses the y-axis at $(0, -\frac{5}{9})$. The graph of $y = f(x)$ is:



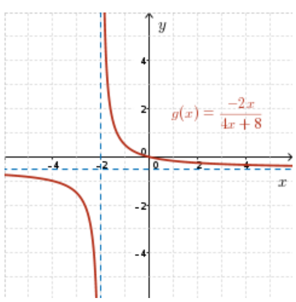
From the graph of $y = f(x)$, we see that the function is positive (above the x-axis) for $\{x \in \mathbb{R} \mid x > 3\}$ and negative (below the x-axis) for $\{x \in \mathbb{R} \mid x < 3\}$. We can also see that $y = f(x)$ is a decreasing function over its domain.

b. The domain of the function $g(x) = \dfrac{-2x}{4x+8}$ is $\{x \in \mathbb{R} \mid x \neq -2\}$ since $4x + 8 \neq 0$.

$g(-2) = \frac{4}{0}$, an undefined value, so the function has a vertical asymptote of $x = -2$. $g(-2.001) = -1000.5$ so $\lim\limits_{x \to -2^-} \dfrac{-2x}{4x+8} = -\infty$ and $g(-1.999) = 999.5$ so $\lim\limits_{x \to -2^+} \dfrac{-2x}{4x+8} = \infty$. As $x \to \pm\infty$, $g(x) \to \dfrac{-2x}{4x}$ so $g(x) \to -\frac{1}{2}$ $\left(\lim\limits_{x \to \pm\infty} \dfrac{-2x}{4x+8} = -\dfrac{1}{2}\right)$ and the horizontal asymptote is $y = -\frac{1}{2}$.

$g(-1000) \approx -0.501 < -\frac{1}{2}$ so the graph approaches $y = -\frac{1}{2}$ from below as $x \to -\infty$ and $g(1000) \approx -0.499 > -\frac{1}{2}$ so the graph approaches $y = -\frac{1}{2}$ from above as $x \to \infty$.

The x-intercept occurs when $g(x) = 0$, that is when $-2x = 0$. Therefore, the function crosses the x-axis at $(0, 0)$. Note that this also implies that the y-intercept occurs at $(0, 0)$. The graph of $y = g(x)$ is:



From the graph of $y = g(x)$, we see that the function is positive (above the x-axis) for $\{x \in \mathbb{R} \mid -2 < x < 0\}$ and negative (below the x-axis) for $\{x \in \mathbb{R} \mid x < -2 \text{ or } x > 0\}$. We can also see that $y = g(x)$ is a decreasing function over its domain.

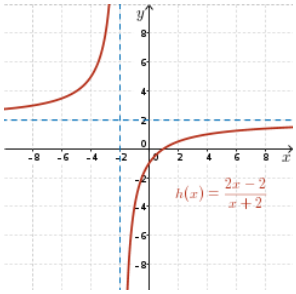
c. The domain of the function $h(x) = \dfrac{2x-2}{x+2}$ is $\{x \in \mathbb{R} \mid x \neq -2\}$ since $x + 2 \neq 0$.

$h(-2) = \dfrac{-6}{0}$, an undefined value, so the function has a vertical asymptote of $x = -2$. $h(-2.001) = 6002$ so $\lim\limits_{x \to -2^-} \dfrac{2x-2}{x+2} = \infty$ and $h(-1.999) = -5998$ so $\lim\limits_{x \to -2^+} \dfrac{2x-2}{x+2} = -\infty$. As $x \to \pm\infty$, $h(x) \to \dfrac{2x}{x}$ so $h(x) \to 2$ $\left(\lim\limits_{x \to \pm\infty} \dfrac{2x-2}{x+2} = 2\right)$ and the horizontal asymptote is $y = 2$.

$h(-1000) \approx 2.006 > 2$ so the graph approaches $y = 2$ from above as $x \to -\infty$ and $h(1000) \approx 1.994 < 2$ so the graph approaches

$y = 2$ from below as $x \to \infty$.

The x-intercept occurs when $h(x) = 0$, that is when $2x - 2 = 0$. Therefore the function crosses the x-axis at $(1, 0)$. The y-intercept occurs when $x = 0$ so the function crosses the y-axis at $(0, -1)$. The graph of $y = h(x)$ is:



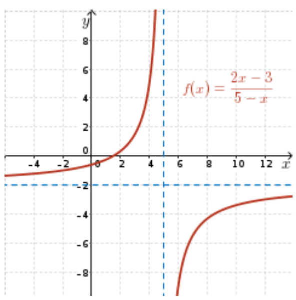
From the graph of $y = h(x)$, we see that the function is positive (above the x-axis) for $\{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$ and negative (below the x-axis) for $\{x \in \mathbb{R} \mid -2 < x < 1\}$. We can also see that $y = h(x)$ is an increasing function over its domain.

d. The domain of $f(x) = \dfrac{2x-3}{5-x}$ is $\{x \in \mathbb{R} \mid x \neq 5\}$ since $5 - x \neq 0$.

$f(5) = \frac{7}{0}$, an undefined value, so the function has a vertical asymptote of $x = 5$. $f(4.999) = 6998$ so $\lim\limits_{x \to 5^-} \dfrac{2x-3}{5-x} = \infty$ and $f(5.001) = -7002$ so $\lim\limits_{x \to 5^+} \dfrac{2x-3}{5-x} = -\infty$. As $x \to \pm\infty, f(x) \to \dfrac{2x}{-x}$ so $f(x) \to -2 \left(\lim\limits_{x \to \pm\infty} \dfrac{2x-3}{5-x} = -2 \right)$ and the horizontal asymptote is $y = -2$.

$f(-1000) \approx -1.993 > -2$ so the graph approaches $y = -2$ from above as $x \to -\infty$ and $f(1000) \approx -2.007 < -2$ so the graph approaches $y = -2$ from below as $x \to \infty$.

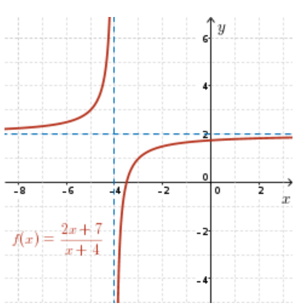
The x-intercept occurs when $f(x) = 0$, that is when $2x - 3 = 0$. Therefore the function crosses the x-axis at $(\frac{3}{2}, 0)$. The y-intercept occurs when $x = 0$ so the function crosses the y-axis at $(0, -\frac{3}{5})$. The graph of $y = f(x)$ is:



From the graph of $y = f(x)$, we see that the function is positive (above the x-axis) for $\{x \in \mathbb{R} \mid \frac{3}{2} < x < 5\}$ and negative (below the x-axis) for $\{x \in \mathbb{R} \mid x < \frac{3}{2} \text{ or } x > 5\}$. We can also see that $y = f(x)$ is an increasing function over its domain.

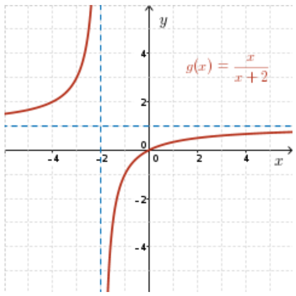
3. There is no solution provided for this question.

4. a. The domain of the function $f(x) = \dfrac{2x+7}{x+4}$ is $\{x \in \mathbb{R} \mid x \neq -4\}$ since $x + 4 \neq 0$. $f(-4) = \dfrac{-1}{0}$, an undefined value, so the function has a vertical asymptote of $x = -4$. As $x \to \pm\infty, f(x) \to \dfrac{2x}{x}$ so $f(x) \to 2 \left(\lim\limits_{x \to \pm\infty} \dfrac{2x+7}{x+4} = 2 \right)$ and the horizontal asymptote is $y = 2$. The x-intercept occurs when $f(x) = 0$, that is, when $2x + 7 = 0$. Therefore the function crosses the x-axis at $(-\frac{7}{2}, 0)$. The y-intercept occurs when $x = 0$ so the function crosses the y-axis at $(0, \frac{7}{4})$.



The domain of the function $g(x) = \dfrac{x}{x+2}$ is $\{x \in \mathbb{R} \mid x \neq -2\}$ since $x + 2 \neq 0$. $g(-2) = \dfrac{-2}{0}$, an undefined value, so the function has a vertical asymptote of $x = -2$. As $x \to \pm\infty, g(x) \to \dfrac{x}{x}$ so $g(x) \to 1$ ($\lim\limits_{x \to \pm\infty} \dfrac{x}{x+2} = 1$) and the horizontal asymptote is $y = 1$. The

x-intercept occurs when $g(x) = 0$, that is, when $x = 0$. Therefore the function crosses the x-axis at $(0, 0)$. The function also crosses the y-axis at $(0, 0)$.



b. To express $f(x) = \dfrac{2x+7}{x+4}$ in the form $y = \dfrac{a}{b(x-h)} + k$, use long division or manipulate the equation algebraically as shown here.

$f(x) = \dfrac{2x+7}{x+4} = \dfrac{2(x+4)-1}{x+4} = \dfrac{2(x+4)}{x+4} + \dfrac{-1}{x+4}$.

Therefore $f(x) = \dfrac{-1}{x+4} + 2$ and the graph of $f(x) = \dfrac{2x+7}{x+4}$ can be obtained by reflecting the graph of $y = \dfrac{1}{x}$ in the x-axis then translating it left 4 units and up 2 units.

To express $g(x) = \dfrac{x}{x+2}$ in the form $y = \dfrac{a}{b(x-h)} + k$, use long division or manipulate the equation algebraically as shown here.

$g(x) = \dfrac{x}{x+2} = \dfrac{(x+2)-2}{x+2} = \dfrac{(x+2)}{x+2} + \dfrac{-2}{x+2}$.

Therefore $g(x) = \dfrac{-2}{x+2} + 1$ and the graph of $g(x) = \dfrac{x}{x+2}$ can be obtained by reflecting the graph of $y = \dfrac{1}{x}$ in the x-axis, applying a vertical stretch from the x-axis by a factor of 2 then translating it left 2 units and up 1 unit.

5. There is no solution provided for this question.

6. a. Since the graph has a vertical asymptote of $x = 2$ then $(x-2)$ must be a factor of the denominator of the rational function, but not the numerator. Working with the form $f(x) = \dfrac{ax+b}{cx+d}$, we have $f(x) = \dfrac{ax+b}{c(x-2)}$. $y = -2$ is a horizontal asymptote. In general, as $x \to \pm\infty$, $f(x) \to \dfrac{a}{c}$. Therefore $\dfrac{a}{c} = -2$. Let $a = -2$, $c = 1$, and $f(x) = \dfrac{-2x+b}{x-2}$. The graph passes through $(0, 0)$ so this point must satisfy our equation.

$$f(x) = \frac{-2x+b}{x-2}$$
$$0 = \frac{-2(0)+b}{0-2}$$
$$\therefore b = 0$$

Therefore, the function $f(x) = \dfrac{-2x}{x-2}$ will satisfy the given conditions.

b. Since the graph has a vertical asymptote of $x = -1$ then $(x+1)$ must be a factor of the denominator of the rational function, but not the numerator. Working with the form $f(x) = \dfrac{ax+b}{cx+d}$, we have $f(x) = \dfrac{ax+b}{c(x+1)}$. $y = 3$ is a horizontal asymptote. In general, as $x \to \pm\infty$, $f(x) \to \dfrac{a}{c}$. Therefore $\dfrac{a}{c} = 3$. Let $a = 3$, $c = 1$, and $f(x) = \dfrac{3x+b}{x+1}$. Since the graph has a y-intercept of $(0, 4)$, this point must satisfy the equation.

$$f(x) = \frac{3x+b}{x+1}$$
$$4 = \frac{3(0)+b}{0+1}$$
$$\therefore b = 4$$

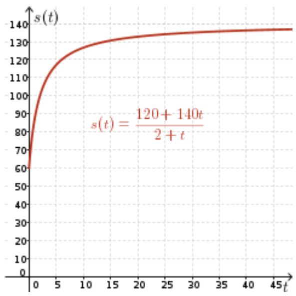
Therefore, the function $f(x) = \dfrac{3x+4}{x+1}$ will satisfy the given conditions.

7. There is no solution provided for this question.

8. a. In the first two hours, the train travels $120\ \text{km}$, then after leaving the construction zone the train travels at $140\ \text{km/h}$ for t hours. Therefore, the total distance travelled is $(120 + 140t)\ \text{km}$ and the total time travelled is $(2 + t)$ hours.

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$
$$s(t) = \frac{120 + 140t}{2 + t}, t \geq 0, t \in \mathbb{R}$$

b. The function is defined for $t \geq 0$ since we cannot have negative time in this situation. At $t = 0$ the average speed is $s(0) = \dfrac{120 + 140(0)}{2 + 0} = 60$. Therefore the y-intercept of the graph is $(0, 60)$. As $t \to \infty$, $s(t) \to \dfrac{140t}{t}$ so $s(t) \to 140$ and the horizontal asymptote is $s = 140$. Finding some specific points we can sketch the graph of the function.



c. There is no vertical asymptote since $2 + t = 0$ when $t = -2$, however, $t \geq 0$. As $t \to \infty$, $s(t) \to \frac{140t}{t}$ so $s(t) \to 140$ and the horizontal asymptote is $s = 140$. The average speed of the train will approach 140 km/h over time but is never able to reach this exact value.

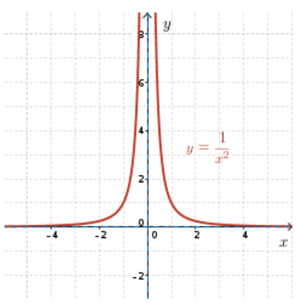
d.

$$\begin{aligned} s(t) &= 100 \\ 100 &= \frac{120 + 140t}{2 + t} \\ 200 + 100t &= 120 + 140t \\ 80 &= 40t \\ \therefore t &= 2 \end{aligned}$$

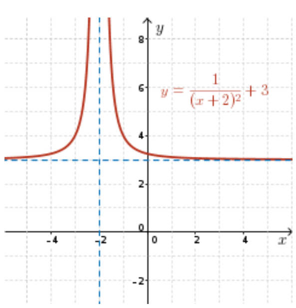
Therefore an average speed of 100 km/h, for the entire trip, is reached 2 hours after leaving the construction zone. The total distance travelled is $120 + 140(2) = 400 \text{ km}$.

9. There is no solution provided for this question.

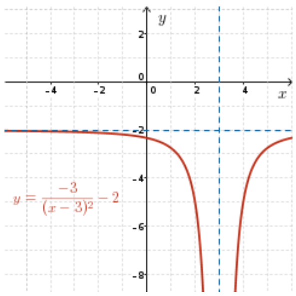
10. a. The domain of the function $y = \frac{1}{x^2}$ is $\{x \in \mathbb{R} \mid x \neq 0\}$. Since $f(0) = \frac{1}{0}$, an undefined value, the function has a vertical asymptote at $x = 0$. As $x \to \pm\infty$, $y \to 0$ so the horizontal asymptote is $y = 0$. The function will be positive for all values of $x, x \neq 0$. A sketch of the function $y = \frac{1}{x^2}$ is:



b. The graph of the function $y = \frac{1}{(x+2)^2} + 3$ can be obtained by translating the graph of $y = \frac{1}{x^2}$, 2 units left and 3 units up. The graph of $y = \frac{1}{(x+2)^2} + 3$ will have a vertical asymptote at $x = -2$ and a horizontal asymptote of $y = 3$. The shape will remain the same.



c. $y = \frac{-3}{x^2 - 6x + 9} - 2 = \frac{-3}{(x-3)^2} - 2$. The graph of $y = \frac{-3}{(x-3)^2} - 2$ can be obtained by reflecting the graph of $y = \frac{1}{x^2}$ in the x-axis, followed by a vertical stretch from the x-axis by a factor of 3, and a translation 3 units right and 2 units down. It will have a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = -2$.



d. First manipulate the equation into the form $y = \dfrac{a}{b(x-h)^2} + k$. Using long division,

$$
\begin{array}{r|rrrrr}
 & & & & & -1 \\
\hline
x^2 - 2x + 1 & & -x^2 & + \ 2x & + & 2 \\
 & - & (-x^2 & + \ 2x & - & 1) \\
\hline
 & & & & & 3
\end{array}
$$

$y = -1 + \dfrac{3}{x^2 - 2x + 1}$ so $y = \dfrac{3}{(x-1)^2} - 1$.The graph of $y = \dfrac{-x^2 + 2x + 2}{x^2 - 2x + 1}$ can be obtained from the graph of $y = \dfrac{1}{x^2}$ by applying a vertical stretch from the x-axis by a factor of 3, followed by a horizontal translation 1 unit to the right and a vertical translation 1 unit down. The graph will have a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = -1$.